Last time:

* Magnetic force on a charge Q:

$$\underline{F} = Q(\underline{v} \times \underline{B})$$

* Magnetic force on a current I:

$$\underline{F} = I\int_a^b d\underline{s} \times \underline{B}$$

* Lorentz force law:

$$\underline{F} = Q[\underline{E} + (\underline{v} \times \underline{B})]$$

Today: Steady currents and the fields they produce

$$\boxed{\begin{array}{lll} \text{stationary charge} \rightarrow & \text{constant } \underline{E} & \text{electrostatics} \\ \text{steady currents} \rightarrow & \text{constant } \underline{B} & \text{magnetostatics} \end{array}}$$

$$\frac{\partial \underline{J}}{\partial t} = 0 \equiv \text{steady current}$$

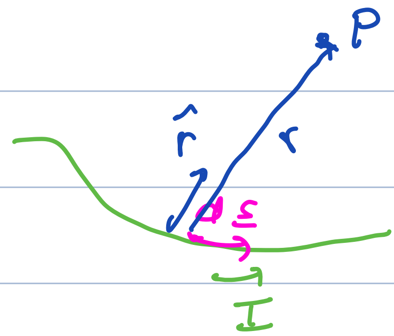
$\underline{J}$ is the current density

→ charge is not piling up $\frac{\partial \rho}{\partial t} = 0$

→ magnitude of the current is the same in all of the wire

→ $\nabla \cdot \underline{J} + \cancel{\frac{\partial \rho}{\partial t}}^{\,0} = 0 \implies \nabla \cdot \underline{J} = 0$

What is $\underline{B}$ of a steady current?



$r \equiv$ distance from source to point P

$\hat{r}$ unit vector in the direction of r

Want to calculate $d\underline{B}$ due to $I d\underline{s}$. Biot Savart law:

$$d\underline{B} = \frac{\mu_0}{4\pi} \frac{I\, d\underline{s} \times \hat{\underline{r}}}{r^2}$$

$\mu_0 \equiv$ permeability of free space $= 4\pi \times 10^{-7}\, T.m/A$

To obtain $\underline{B}$ we will integrate $d\underline{B}$ over the wire:

$$\boxed{\underline{B} = \int_{wire} d\underline{B} = \frac{\mu_0 I}{4\pi} \int_{wire} \frac{d\underline{s} \times \hat{r}}{r^2}}$$

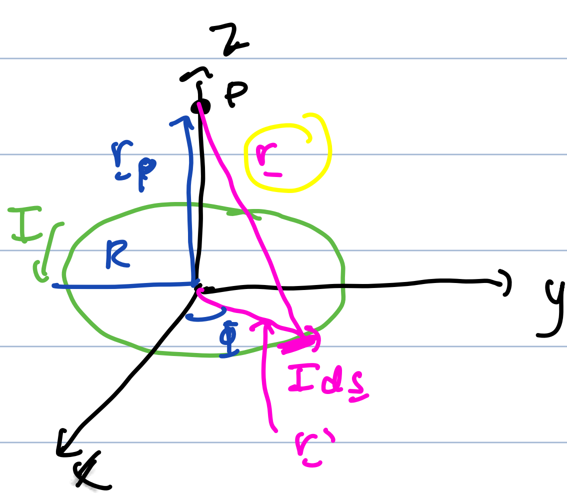
$\hat{r}$ is the unit vector in the direction of r

distance from the current source to the field point

$\underline{r} = r\hat{r}$

$\underline{B}$ has 3 components. Expression for $\underline{B}$ is really 3 integrals.

Example: What is $\underline{B}$ of a circular current loop?



$$\underline{B} = \frac{\mu_0 I}{4\pi} \int_{wire} \frac{d\underline{s} \times \hat{r}}{r^2}$$

General method to solve this problems:

1) Source point: Write an expression for $I d\underline{s}$ and the vector $\underline{r}'$ describing its position will respect to the coord system

2) Field point: It's the point where we want to calculate the field. Its position will be given by $\underline{r}_p$

3) Relative position vector: Vector between the source point and field point

$$\underline{r} = \underline{r}_p - \underline{r}'$$

$$\hat{r} = \frac{\underline{r}}{r} = \frac{\underline{r}_p - \underline{r}'}{|\underline{r}_p - \underline{r}'|}$$

4) Find what these vectors are because to calculate $d\underline{s} \times \vec{r}$ which will give the direction of $\underline{B}$

5) Once we have this, we just need to subs into Biot-Savart's law and simplify.

1) Find the source point $\underline{r}'$

$\underline{r}' = R(\cos\Phi\,\hat{i} + \sin\Phi\,\hat{j})$. With this we can write

$$I d\underline{s} = I\left(\frac{d\underline{r}'}{d\Phi'}\right) d\Phi' = I\, d\Phi' \frac{d}{d\Phi'}\left[R(\cos\Phi'\,\hat{i} + \sin\Phi'\,\hat{j})\right]$$

$$= IR\,d\Phi'(-\sin\Phi'\,\hat{i} + \cos\Phi'\,\hat{j})$$

2) What about the field point?

$\underline{r}_p = z\hat{k}$ a point in the z axis

3) What is the relative position vector?

$$\underline{r} = \underline{r}_p - \underline{r}' = -R\cos\Phi'\,\hat{i} - R\sin\Phi'\,\hat{j} + z\hat{k}$$

We can find its magnitude:

$$r = |\underline{r}| = \left[(-R\cos\Phi')^2 + (-R\sin\Phi')^2 + z^2\right]^{1/2}$$

$\sin^2\Phi' + \cos^2\Phi' = 1$

$$= \left[R^2 + z^2\right]^{1/2}$$

We can write the unit vector as $\hat{r} = \frac{\underline{r}}{|r|}$

4) Calculate the cross product $d\underline{s} \times \vec{r}$

$$d\underline{s} \times \underbrace{(\underline{r}_p - \underline{r}')}_{\underline{r}} = R\,d\Phi'(-\sin\Phi'\vec{i} + \cos\Phi'\vec{j}) \times$$

$$[-R\cos\Phi'\vec{i} - R\sin\Phi'\vec{j} + z\vec{k}]$$

$$= R\,d\Phi'[z\cos\Phi'\vec{i} + z\sin\Phi'\vec{j} + R\vec{k}]$$

5) Write down $d\underline{B}$

$$d\underline{B} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s}\times\vec{r}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s}\times\underline{r}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{d\underline{s}\times(\underline{r}_p - \underline{r}')}{|\underline{r}_p - \underline{r}'|^3}$$

$$= \frac{\mu_0 I R}{4\pi}\frac{z\cos\Phi'\vec{i} + z\sin\Phi'\vec{j} + R\vec{k}}{(R^2+z^2)^{3/2}}\,d\Phi'$$

6) Integrate:

$$\underline{B} = \frac{\mu_0 I R}{4\pi}\int_0^{2\pi}\frac{z\cos\Phi'\vec{i} + z\sin\Phi'\vec{j} + R\vec{k}}{(R^2+z^2)^{3/2}}\,d\Phi'$$

We will have 3 integrals:

$$B_x = \frac{\mu_0 I R z}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\cos\Phi'\,d\Phi' = \frac{\mu_0 I R z}{4\pi(R^2+z^2)^{3/2}}\sin\Phi'\Big|_0^{2\pi} = 0$$

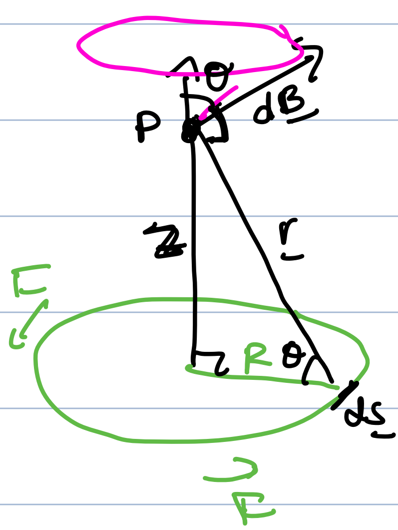
$$B_y = \frac{\mu_0 I R z}{4\pi(R^2+z^2)^{3/2}}\int_0^{2\pi}\sin\Phi'\,d\Phi' = \frac{-\mu_0 I R z}{4\pi(R^2+z^2)^{3/2}}\cos\Phi'\Big|_0^{2\pi} = 0$$

$$B_z = \frac{\mu_0}{4\pi} \frac{IR^2}{(R^2+z^2)^{3/2}} \int_0^{2\pi} d\Phi' = \frac{\mu_0}{4\pi} \frac{2\pi IR^2}{(R^2+z^2)^{3/2}}$$

$$= \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$$

$$\therefore \boxed{B_z(z) = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}}$$

Noticing before some symmetries we can arrive at the solution faster



* $d\underline{s} \times \vec{r}$ defines the direction of $\underline{B}$, the direction of $\underline{B}$ should be ⊥ to $d\underline{s}$ and $\vec{r}$

* As we integrate around the circle the vector $d\underline{B}$ will sweep a cone.

* Because of symmetry horizontal components of dB will cancel, leaving only z.

$$dB = \frac{\mu_0 I}{4\pi} \frac{d\underline{s} \times \vec{r}}{r^2} = \frac{\mu_0 I}{4\pi} \frac{ds \cos\theta}{r^2}$$

$$B(z) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{ds \cos\theta}{r^2} = \frac{\mu_0}{4\pi} I \int \frac{ds}{r^2} \cos\theta = \frac{\mu_0}{4\pi} \frac{I\cos\theta}{r^2} \int_0^{2\pi} ds$$

$$= \frac{\mu_0}{4\pi} \frac{I\cos\theta}{r^2} \underbrace{2\pi R}_{\text{perimeter}} = \frac{\mu_0 IR}{2r^2} \Big(\underbrace{\frac{R}{r}}_{\cos\theta \text{ by trigonometry}} \Big)$$

$$= \frac{\mu_0 IR^2}{\underbrace{2r^3}_{\text{pythagoras}}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$$

We can generalize Biot-Savart's law for surface and steady currents:

$$\mathbf{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{K}(\underline{r}) \times \hat{r}}{r^2}\, dA \qquad \text{surface current}$$

$$\mathbf{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{r}) \times \hat{r}}{r^2}\, d\tau \qquad \text{volume currents.}$$

$\underline{r} = |\underline{r}|\hat{r}$ is the vector from the source of B to the point of interest.

* The superposition principle applies to magnetic fields just as for electric fields

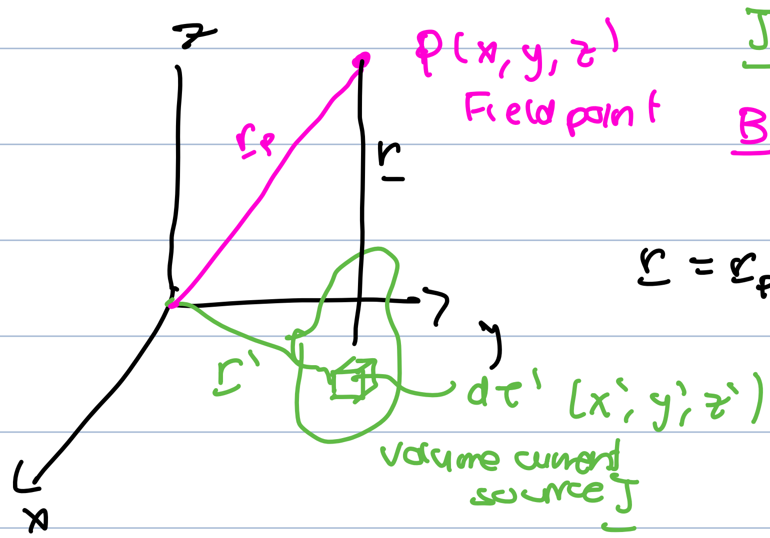
Net $\underline{B}$ is the sum of the fields from all sources

* We cannot write the field of a moving charge using B-S law:

$$\cancel{\mathbf{B}(\underline{r}) = \frac{\mu_0}{4\pi} \frac{q\,\underline{v} \times \hat{r}}{r^2}}$$ point charge is NOT a steady current.

The divergence and the curl of $\underline{B}$

Ampere's law



$\underline{J}$ is a function of (x', y', z')

$\underline{B}$ is a function (x, y, z)

$$\underline{r} = \underline{r}_p - \underline{r}' = (x - x')\hat{x} + (y - y')\hat{y} + (z - z')\hat{z}$$

Biot-Savart's law:

$$\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int \frac{\underline{J}(\underline{r}')\times\hat{r}}{r^2}\,d\tau'$$

Using the identity:

$$\nabla\cdot\left(\underline{J}\times\frac{\hat{r}}{r}\right) = \frac{\hat{r}}{r^2}\cdot(\nabla\times\underline{J}) - \underline{J}\cdot\left(\nabla\times\frac{\hat{r}}{r^2}\right)$$

p. 21 Griffiths product rules

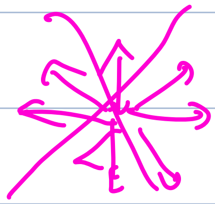
We can subs this in the eq. for the div. of $\underline{B}$:

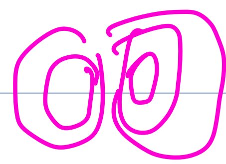
$$\nabla\cdot\underline{B} = \frac{\mu_0}{4\pi}\int\left[\frac{\hat{r}}{r^2}\cdot\cancel{(\nabla\times\underline{J})}^{\,0} - \underline{J}\cdot\cancel{\left(\nabla\times\frac{\hat{r}}{r^2}\right)}^{\,0}\right]$$

because $\underline{J}$ is indep of the unprimed coord.

$$\boxed{\nabla\cdot\underline{B} = 0}$$

Divergence of the magnetic field is zero
Non-existence of magnetic monopoles.





magnetic field lines form closed loops or extend ∞

Now we'll calculate the curl:

$$\nabla\times\underline{B}(\underline{r}) = \frac{\mu_0}{4\pi}\int\nabla\times\left(\underline{J}\times\frac{\hat{r}}{r^2}\right)d\tau'$$

We will expand the integrand using the following identity:

$$\nabla\times\left(\underline{J}\times\frac{\hat{r}}{r^2}\right) = \underline{J}\left(\nabla\cdot\frac{\hat{r}}{r^2}\right) - \cancel{(\underline{J}\cdot\nabla)\frac{\hat{r}}{r^2}} - \frac{\hat{r}}{r^2}\cancel{(\nabla\cdot\underline{J})} + \cancel{\left(\frac{\hat{r}}{r^2}\cdot\nabla\right)\underline{J}}^{\,0}$$

product rule (vi) Griffiths p. 21

because of

(integrate to zero)

the derivative w/ respect to unprimed coord.

$$\underline{J} \left[\nabla \cdot \frac{\hat{\underline{r}}}{r^2}\right] = 4\pi \delta^3(\underline{r})$$ see p.50 Griffiths.

With this we have the curl of $\underline{B}$:

$$\nabla \times \underline{B} = \frac{\mu_0}{\cancel{4\pi}} \int \underline{J}(\underline{r}')\, \cancel{4\pi}\, \delta^3(\underline{r}-\underline{r}')\, d\tau' = \mu_0 \underline{J}(\underline{r})$$

$$\boxed{\nabla \times \underline{B} = \mu_0 \underline{J}}$$ Ampère's law in differential form

We can use Stoke's theorem to convert it into integral form

$$\int (\nabla \times \underline{B}) \cdot d\underline{a} \overset{\uparrow\ \text{Stokes theorem}}{=} \oint \underline{B} \cdot d\underline{l} = \mu_0 \underbrace{\int \underline{J} \cdot d\underline{a}}_{\text{total current passing through a surface}} = \mu_0 I$$

$I = \int \underline{J} \cdot d\underline{a}$ by definition

$$\boxed{\oint \underline{B} \cdot d\underline{l} = \mu_0 I_{enc}}$$ Ampère's law in integral form

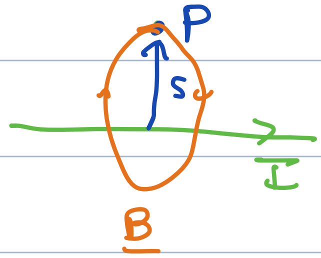
μ_0 = permittivity of free space

I_{enc} = current enclosed by an amperian loop.

The direction of integration around the boundary is given by the right hand rule.

Your fingers of the right hand indicate the direction of integration, your thumb defines the direction of pos current.

Example: Calculate $\underline{B}$ a distance s from a long straight wire carrying steady current I



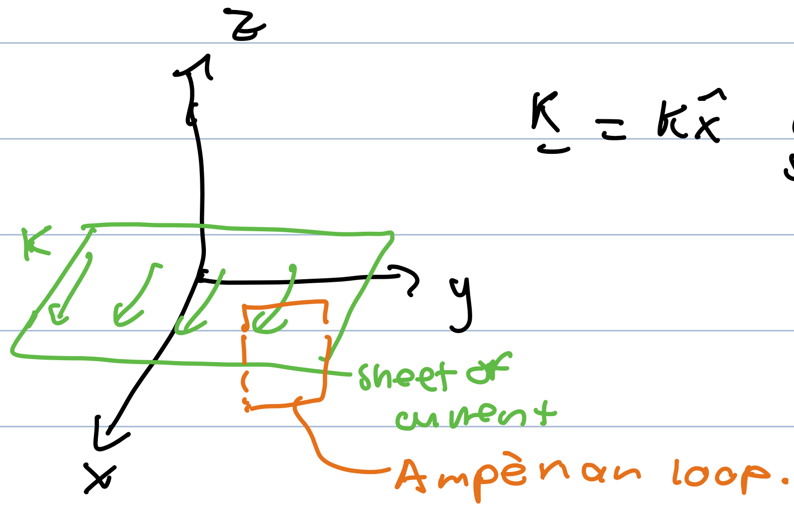
$$\oint \underline{B}\cdot d\underline{l} = B \oint dl = (2\pi s) = \mu_0 I_{enc}$$

circling around wire using right hand rule

B is constant around the loop

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi s}}$$

Example: Calculate the magnetic field of an infinite uniform surface current.



$\underline{K} = K\hat{x}$ current flowing in a sheet over xy plane

What is the direction of $\underline{B}$?

$$B(\underline{r}) = \frac{\mu_0}{4\pi} \int \frac{\underline{K}(\underline{r}) \times \hat{r}}{r^2} \, d\underline{a}$$

x component is zero in the cross product

The field is ⊥ to K $\Rightarrow$ its perpendicular to x

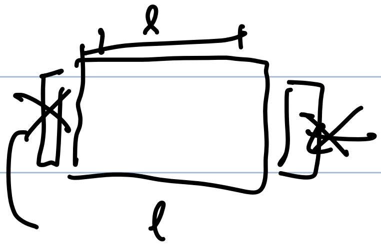
Can the field have a z component?

No because of symmetry $\underline{K}$ is symmetric with respect to z axis.

Now we'll look at the loop :

$$\oint \underline{B}\cdot d\underline{l} = B(2l) = \mu_0 I_{enc} = \mu_0 K l$$

definition K



$\underline{B}\cdot d\underline{l} = 0$

$B \perp dl$

$$\Rightarrow B = \frac{\mu_0 K}{2}$$

$$B = \begin{cases} \frac{\mu_0}{2} K \hat{y}, & z < 0 \\ -\frac{\mu_0}{2} K \hat{y}, & z > 0 \end{cases}$$